Available online at www.sciencedirect.com

ScienceDirect

AKCE International Journal of Graphs and Combinatorics 13 (2016) 177–190

AKCE International Journal of Graphs and Combinatorics

www.elsevier.com/locate/akcej

A number theoretic problem on super line graphs

Jay Bagga[a,*], Lowell Beineke[b], Badri Varma[c]

[a] *Ball State University, Muncie, IN 47306, USA*
[b] *Indiana University - Purdue University Fort Wayne, Fort Wayne, IN 46805, USA*
[c] *University of Wisconsin - Fox Valley, Menasha, WI 54952, USA*

Received 22 February 2016; accepted 4 May 2016
Available online 11 June 2016

Abstract

In Bagga et al. (1995) a generalization of the line graph concept was introduced. Given a graph G with at least r edges, the *super line graph of index* r, $\mathcal{L}_r(G)$, has as its vertices the sets of r edges of G, with two adjacent if there is an edge in one set adjacent to an edge in the other set. The *line completion number* $\mathrm{lc}(G)$ of a graph G is the least index r for which $\mathcal{L}_r(G)$ is complete. In this paper we investigate the line completion number of $K_{m,n}$. This turns out to be an interesting optimization problem in number theory, with results depending on the parities of m and n. If $m \leq n$ and m is a fixed even number, then $\mathrm{lc}(K_{m,n})$ has been found for all even values of n and for all but finitely many odd values. However, when m is odd, the exact value of $\mathrm{lc}(K_{m,n})$ has been found in relatively few cases, and the main results concern lower bounds for the parameter. Thus, the general problem is still open, with about half of the cases unsettled.

Keywords: Line graph; Super line graph; Line completion number

1. Introduction

The line graph of a graph is arguably the most important and best loved of graph operations, and so not surprisingly mathematicians have taken pleasure in generalizing it in many ways. In forming the line graph of a graph G, one takes the individual edges of G as the vertices of the new graph $L(G)$, with two joined if they have a common vertex, as illustrated in Fig. 1.

In most generalizations of line graphs, the vertices of the new graph are taken to be another family of subgraphs of G, and adjacency is defined in terms of an appropriate intersection. For example, in the r-path graph $\mathcal{P}_r(G)$, the vertices are the paths in G of length r, with adjacency being defined as overlapping in a path of length $r - 1$. Fig. 2 shows the 2-path graph operation.

In Bagga, Beineke, and Varma [1], a different choice was made, one that has turned out to yield many interesting results. Given a graph G with at least r edges, the *super line graph* (*of index* r) $\mathcal{L}_r(G)$ has as its vertices the sets of r

Peer review under responsibility of Kalasalingam University.
* Corresponding author.
E-mail addresses: jbagga@bsu.edu (J. Bagga), beineke@ipfw.edu (L. Beineke), badri.varma@uwc.edu (B. Varma).

http://dx.doi.org/10.1016/j.akcej.2016.05.001
0972-8600/

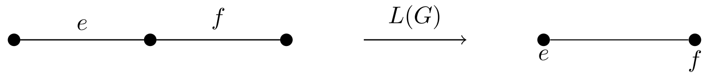

Fig. 1. The line graph operation.

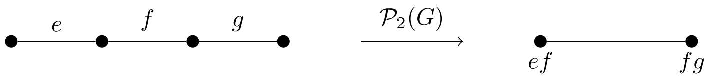

Fig. 2. The path graph operation.

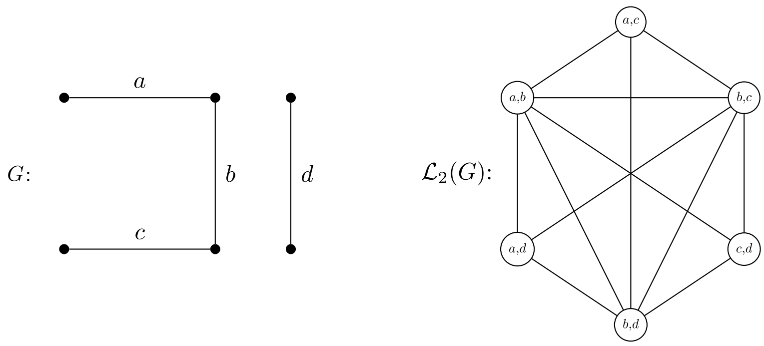

Fig. 3. An example of a super line graph of index 2.

edges of G, with two adjacent if there is an edge in one set adjacent to an edge in the other set. An example of a super line graph of index 2 is shown in Fig. 3.

For convenience, in the remainder of this article we adopt the convention that when we speak of a super line graph $\mathcal{L}_r(G)$ of index r, the base graph G has at least r edges. Other notation conventions that we adopt are the following: $\|G\|$ denotes the *edge-order* of G (the number of edges in the graph G); $\langle A\rangle$ denotes the subgraph induced by a nonempty subset A of the vertices of G; and $G + H$ denotes the union of disjoint graphs G and H, with kG denoting the graph consisting of k copies of G.

1.1. Index 2

Not surprisingly, the super line graphs of index 2 have attracted the greatest interest, and in this subsection, we give a brief survey of some of the results on this family, beginning with a couple of observations [2] about two pairs of edges S and T in a graph G that are not adjacent in $\mathcal{L}_2(G)$:

(a) In the graph of the union of the two pairs, the intersection is either empty or an isolated edge.
(b) If neither S nor T is a set of two isolated edges in G, then the distance between them in $\mathcal{L}_2(G)$ is 2. (To see this, choose a pair R in $\mathcal{L}_2(G)$ that contains one edge adjacent to one in S and one edge adjacent to one in T; then R is adjacent to both S and T.)

Consequently, we have the following result.

Theorem 1.1. *If G has at most one isolated edge, then the diameter of $\mathcal{L}_2(G)$ is* 1 *or* 2*.*

Some other interesting results on this family of graphs involve Hamiltonian properties; for example, it has been shown that (see Bagga, Beineke, and Varma [3] and Li, Li, and Zhang [4]) if G is connected, then $\mathcal{L}_2(G)$ is vertex-pancyclic and path-comprehensive (that is, every pair of vertices are joined by paths of all lengths greater than 1).

One area of investigation in the study of line graphs themselves is to find all those graphs whose line graph has a specified property. For example, in considering the property of completeness, the only connected graphs whose line graphs are complete are the stars $K_{1,s}$ and the complete graph K_3. The following result [2] tells when the super line graph of index 2 is complete.

Theorem 1.2. *The super line graph of index* 2 *of a connected graph with at least two edges is complete if and only if it does not contain either* $3K_2$ *or* $2K_{1,2}$ *as a subgraph.*

Of course, most graphs contain such a subgraph; for instance, any graph having a path of length 5 does. However, as we shall see, for any graph there are indices for which the super line graph is complete, the topic we turn to next.

1.2. The line completion number

This subsection is devoted to the topic that gave rise to our consideration of the subject of the paper: the line completion number of a graph. Before getting to that, we provide some general results on subgraphs of super line graphs (see [1] and [5]).

Theorem 1.3. *For all graphs G and all $r \geq 2$, $\mathcal{L}_r(G)$ has at most one nontrivial component.*

Theorem 1.4. *If H is a subgraph of G with at least r edges, then $\mathcal{L}_r(H)$ is an induced subgraph of $\mathcal{L}_r(G)$.*

Theorem 1.5. *If G is a graph with m edges, then for $r < m/2$, $\mathcal{L}_r(G)$ is isomorphic to a subgraph of $\mathcal{L}_{r+1}(G)$.*

We noted earlier that unless two components of a graph G are single edges, then $\mathcal{L}_2(G)$ has diameter 1 or 2, and so, loosely speaking, it does not "spread out" much. A similar statement holds for higher indices: If no more than $r-1$ components of G are single edges, then $\mathcal{L}_r(G)$ has diameter 1 or 2.

Theorem 1.6. *For any graph G, if $\mathcal{L}_r(G)$ is complete, then so is $\mathcal{L}_{r+1}(G)$.*

It follows from this result that there is a least index for which the super line graph of a graph is complete, and thus all those of greater index are also complete. Formally, we have this definition: The *line completion number* $\mathrm{lc}(G)$ of a graph G is the least index r for which $\mathcal{L}_r(G)$ is complete.

The first of the next two results (see [5]) gives those graphs whose line completion number is small, while the second gives those for which it is as large as possible.

Theorem 1.7. *Let G be a graph.*

- $\mathrm{lc}(G) = 1$ *if and only if G is K_3 or $K_{1,s}$.*
- $\mathrm{lc}(G) \leq 2$ *if and only if G does not have $3K_2$ or $2P_3$ as a subgraph.*
- $\mathrm{lc}(G) \leq 3$ *if and only if G does not have any of seven known graphs as a subgraph.*

Theorem 1.8. *Let G be a graph with m edges.*

- $\mathrm{lc}(G) = m$ *if and only if G is mK_2.*
- $\mathrm{lc}(G) = m-1$ *if and only if G is $P_3 + (m-2)K_2$ or $2P_3 + (m-4)K_2$.*

The following theorem gives a sharp upper bound for the line completion number in terms of the order and number of components.

Theorem 1.9. *If G is a graph with m edges and c components, then*

$$\mathrm{lc}(G) \leq \left\lfloor \frac{m+c}{2} \right\rfloor.$$

Furthermore, this bound is sharp for all m and c.

In the remainder of the paper, we are interested only in connected graphs, and for these we have this result:

Corollary 1. *If G is a connected graph with m edges, then*

$$\mathrm{lc}(G) \leq \left\lceil \frac{m}{2} \right\rceil.$$

This bound is sharp for all m.

Turning to lower bounds, as we noted earlier, if S and T are two sets of r edges having no vertices in common, then the corresponding vertices are not adjacent in $\mathcal{L}_r(G)$. Consequently, we have the next result.

Lemma 1. *For any graph G,*

$$\mathrm{lc}(G) \geq 1 + \max\{\|\langle A\rangle\|\},$$

where the maximum is over all sets $A \subset V$ for which $\|\langle A\rangle\| \leq \|\langle V \setminus A\rangle\|$.

These two bounds have been very useful in establishing other results. For example, we have the following theorem on trees.

Theorem 1.10. *If T is a tree of order $n \geq 2$, then $\mathrm{lc}(T) \leq \lfloor \frac{n}{2} \rfloor$. Furthermore, for any integer k satisfying $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, there exists a tree of order n and line completion number k.*

The bounds have also been useful in determining the line completion number of the graphs in several common families of graphs. Here, as usual, P_n and C_n denote the path and cycle of order n, respectively. Also, the fan F_n, the wheel W_n, and the windmill M_n are the graphs resulting from adding one vertex adjacent to all of the vertices in the path P_n, the cycle C_n, and the graph nK_2, respectively.

Theorem 1.11.
- *Complete graphs: $\mathrm{lc}(K_n) = \binom{p}{2} + 1$, where $p = \lfloor \frac{n}{2} \rfloor$;*
- Paths and cycles*: $\mathrm{lc}(P_n) = \mathrm{lc}(C_n) = \lfloor \frac{n}{2} \rfloor$;*
- Fans and wheels*: $\mathrm{lc}(F_n) = \mathrm{lc}(W_n) = \lfloor \frac{2n}{3} \rfloor$;*
- Windmills*: $\mathrm{lc}(M_n) = \lfloor \frac{3n}{4} \rfloor + 1$.*

Notably absent from these results is the family of complete bipartite graphs $K_{m,n}$, — and that problem is the subject of this paper. Because of the structure of complete bipartite graphs, it turns out the lower bound of Lemma 1 is important. Since every induced subgraph of a complete bipartite graph is also complete bipartite, this observation can be restated as follows:

$$\mathrm{lc}(K_{m,n}) = 1 + \max\{ab : a \leq m, b \leq n, ab \leq (m-a)(n-b)\}.$$

It is therefore useful to have this notation:

$$\beta(m, n) = \max\{\min\{ab, (m-a)(n-b)\}\},$$

where the maximum is taken over all a and b with $0 \leq a \leq m$ and $0 \leq b \leq n$.

Thus, since $\mathrm{lc}(K_{m,n}) = 1 + \beta(m, n)$, the line completion number of complete bipartite graphs boils down to a purely combinatorial problem, and that is the approach we take here. Hence, in the remainder of the paper, the problem that we concentrate on is simply this:

Find $\beta(m, n)$ for positive integers m and n.

We first considered this problem some years ago, and realized then that it was unlikely to have a simple solution. Gutiérrez and Lladó [6] showed that for all r and s, $\beta(2r, 2s) = rs$ (which we had also observed) and conjectured that $\beta(2r, 2s+1) = rs$ and that $\beta(2r+1, 2s+1) = r(s+1)$ when $r \leq s$. As we show, when one of m and n is odd and the other even, their conjecture does not always hold. However, for a given even value of m, this is the case for only a finite number of odd values of n. On the other hand, when m and n are both odd, their conjecture almost never holds, and in fact can be off by an arbitrarily large amount.

2. Results

It is not surprising that the answer to our question depends on the parities of m and n, and we consider separately the four possible combinations of odd and even values of m and n with $m \leq n$:

- Type I: Both m and n are even.
- Type II: Both m and n are odd.
- Type III: m is even and n is odd.
- Type IV: m is odd and n is even.

While the first case is simple (we show that $\beta(m,n) = \frac{1}{4}mn$), the other three are more complicated, and therefore more interesting, and in fact we have been unable to find a complete solution in any of those cases. One noteworthy difference between them is what happens in the long run. If m is even and n is odd, then (as we shall show) for fixed m, $\beta(m,n) = \frac{1}{4}m(n-1)$ holds for all but finitely many values of n. However, if both are odd, then there is no comparable formula.

Before considering the four types individually, we introduce a bit more notation and terminology. Given m and n, we let $r = \lfloor \frac{m}{2} \rfloor$ and $s = \lfloor \frac{n}{2} \rfloor$ (thus, $m = 2r$ or $2r+1$ and $n = 2s$ or $2s+1$, depending on their parity). We define the *base case* for each of the four types:

- Type I: For $m = 2r$ and $n = 2s$, the *base case* is the pair (r, s).
- Type II: For $m = 2r+1$ and $n = 2s+1$ with $r \le s$, the *base case* is the pair $(r, s+1)$.
- Type III: For $m = 2r$ and $n = 2s+1$, the *base case* is the pair (r, s).
- Type IV: For $m = 2r+1$ and $n = 2s$, the *base case* is again the pair (r, s).

A pair (a, b) is called *optimal* if $ab = \beta(m,n)$ where $0 \le a \le m$ and $0 \le b \le n$.

It is easily seen that within each of the four types the product of the two numbers in the base case gives a lower bound for $\beta(m,n)$. One question that we consider is when does equality hold; that is, when is the base case optimal.

2.1. *Type I*

The solution for this type is straightforward: we simply split each number in half.

Theorem 2.1. *If $m = 2r$ and $n = 2s$, then $\beta(m,n) = rs$.*

Proof. If $\beta(m,n) > rs$, then there must exist $x \ge 0$, $y \ge 0$ for which both $(r+x)(s-y) > rs$ and $(r-x)(s+y) > rs$, and this is easily seen to be impossible. ■

2.2. *Type II*

We now turn to the situation with both m and n odd, $m = 2r+1$ and $n = 2s+1$ with $r \le s$, and the base case is the pair $(r, s+1)$ with product $r(s+1)$. Most of our results in this section concern the question of whether the base case holds or not. In general, when the base case does not hold, finding the exact value seems difficult. Therefore, our results usually involve either finding a split of m and n that does better than the base case or showing that no better split exists.

We observe that when the base case does not hold, that is, when $\beta(m,n) > r(s+1)$, there exist non-negative integers x and y for which both $(r-x)(s+1+y)$ and $(r+1+x)(s-y)$ exceed $r(s+1)$:

$$(r-x)(s+1+y) \ge rs + r + 1 \tag{1}$$

and

$$(r+1+x)(s-y) \ge rs + r + 1. \tag{2}$$

It follows from (1) that

$$y \ge \frac{x(s+1)+1}{r-x}, \tag{3}$$

and similarly from (2) that

$$y \le \frac{(x+1)s - r - 1}{r+x+1}. \tag{4}$$

We note that if $x = 0$, then, taken together, (3) and (4) imply that $1 \le y \le \frac{s-r-1}{r+1}$, so $s \ge 2r+2$. It is convenient to split this type into two sub-types along these lines; that is,

Type II(a): $r \le s \le 2r+1$,

Type II(b): $s \ge 2r+2$.

In terms of m and n, in Type II(a), $m \le n \le 2m+1$, and in Type II(b), $n \ge 2m+3$.

Table 1
Several values for Type II(a): Both m and n odd, $m \le n \le 2m+1$.

r	s	$m = 2r+1$	$n = 2s+1$	Base value $= r(s+1)$	$\beta(m,n)$
13	21	27	43	286	286
13	22	27	45	299	**300**
14	23	29	47	336	336
17	28	35	57	493	**494**
17	29	35	59	510	**512**

2.2.1. *Type II(a)*

Table 1 shows that sometimes the base case holds and sometimes it does not. The values for $\beta(m,n)$ in Table 1 were determined with a computer program that we developed. The values in **bold** are those where the base case does not hold. Our computer program verified that the smallest pair (m,n) of Type II(a) for which the base case is not optimal has $r = 13$ and $s = 22$, for which $\beta(27,45) = 300$. For the remainder of Section 2.2.1 we assume that $r \ge 13$. (In Section 2.2.2 we will prove that the base case does not hold for Type II(b).)

We first show that for given m there are five values of n at the upper end of the range for which the base case is optimal.

Theorem 2.2. *If $m = 2r+1$ and $n = 2s+1$ and $d = s-r$, then for $r-3 \le d \le r+1$, the base case holds, that is, $\beta(m,n) = r(s+1)$. Furthermore, these bounds on d are sharp.*

Proof. Suppose not. For convenience, we let $a = s-2r$ (so $d = r+a$). Then it follows from (3) that

$$y \ge \frac{2rx+ax+x+1}{r-x} = 2x + \frac{2x^2+(a+1)x+1}{r-x},$$

and since the last fraction is positive for $a \ge -3$ and y must be an integer,

$$y \ge 2x+1.$$

Likewise, (4) implies that

$$y \le \frac{r(2x+1)+ax+a-1}{r+x+1} = 2x+1-\frac{2x^2+(3-a)x+(2-a)}{r+x+1},$$

and so, by similar reasoning, since $a \le 1$,

$$y \le 2x.$$

Since the two bounds on y are inconsistent, the result follows.

Now consider the case $s = 2r-4$. It is straightforward to check that both of the products $(r-1)(s+3)$ and $(r+2)(s-2)$ exceed $r(s+1)$ (in the above notation, $x = 1$ and $y = 2$). Thus, the base case does not hold. Similarly, for $s = 2r+2$, the products $r(s+2)$ and $(r+1)(s-1)$ exceed $r(s+1)$ (here $x = 0$ and $y = 1$). Thus, both bounds are sharp. ■

We note that the base case also holds when m is relatively close to n.

Theorem 2.3. *With $13 \le r \le s \le 2r+1$ and $d = s-r$, if $d \le 5 + \lceil \frac{27}{r-4} \rceil$, then the base case holds.*

Proof. Suppose not. Then there exist x and y satisfying (3) and (4). Since $r \le s \le 2r+1$, it follows from our remark above that $x \ge 1$. From (3) and (4) it follows that

$$\frac{x(s+1)+1}{r-x} \le \frac{s(x+1)-r-1}{r+x+1}.$$

Letting $s = r+d$, we get, upon simplification,

$$(2r+1)(x^2+x+1) \le d(r-(2x^2+2x)).$$

Table 2
Several values for Type II(a) for $r = 100$.

r	s	Optimum
100	100–136	Base case
	137–145	$x = 2, y = 3$
	146–152	Base case
	153–177	$x = 1, y = 2$
	178–185	$x = 2, y = 4$
	186–188	$x = 3, y = 6$
	189–192	$x = 2, y = 4$
	193–196	$x = 1, y = 2$
	197–201	Base case

It follows that since $x \geq 1$,

$$d \geq \frac{3(2r+1)}{r-4},$$

which contradicts the bound on d in the hypothesis. ■

We now look a bit further at what this theorem tells us about the greatest value $f(r)$ of s for which the base case holds for $r, r+1, \ldots, s$. Our data show that $f(r) = 2r+1$ for $r \leq 12$. However, $f(13) = 21$, and, as noted earlier, the smallest pair (m, n) of Type II(a) for which the base case is not optimal has $r = 13$ and $s = 22$, for which $\beta(27, 45) = 300$. In fact, as a special case of the next theorem, we see that the base case is not optimal for $r \geq 13$ and $\frac{3r+5}{2} \leq s \leq 2r-4$.

Based on small values (up to $r = 40$), we thought it reasonable to conjecture that the least value of s for which the base case does not hold has $x = 1$ and $y = 2$, whence $s = \lceil \frac{3r+4}{2} \rceil$. However, this is not the case, with the least counterexample being for $r = 47$ and $s = 66$, with $x = 2$ and $y = 3$. (It is interesting that there is no similar value for $r = 48$.)

Table 2 lists several values for $r = 100$. This leads us to believe that the situation is not simple:

Our data also show that, for a fixed value of r, with $13 \leq r \leq 200$, the base case does not hold for several ranges of values of s, with $r \leq s \leq 2r+1$. Our next theorem shows this to be indeed the case in general.

Theorem 2.4. *For a fixed r, let $z \geq 1$ be the largest integer that satisfies $r \geq 2z^3 + 5z^2 + 5z + 1$. Then, for $1 \leq u \leq z$, and for*

$$\frac{(u+2)r + (u+1)^2 + 1}{u+1} \leq s \leq \frac{(u+1)r - (u+1)^2}{u},$$

the base case does not hold; that is, $\beta(2r+1, 2s+1) > r(s+1)$.

Proof. Since

$$\frac{(u+1)r - (u+1)^2}{u} - \frac{(u+2)r + (u+1)^2 + 1}{u+1} = \frac{r - (2u^3 + 5u^2 + 5u + 1)}{u(u+1)},$$

the inequalities in the statement of the theorem are well-defined.

We show that the split of $2r+1$ into $r-u$ and $r+u+1$ and that of $2s+1$ into $s+u+2$ and $s-u-1$ (correspondingly) gives both of the relevant products exceeding that of the base case.

First, suppose this is not so for the first product; that is,

$$(r-u)(s+u+2) \leq r(s+1).$$

It follows that

$$s > \frac{(u+1)r - (u+1)^2}{u},$$

which violates the upper bound on s in the hypothesis, and so the first inequality is satisfied.

Similarly, suppose that

$$(r+u+1)(s-u-1) \leq r(s+1).$$

Then

$$s < \frac{(u+2)r + (u+1)^2 + 1}{u+1},$$

which violates the lower bound on s.

Consequently, the base case does not hold for the given values of r and s. ■

With the notation as in Theorem 2.4, when, for example, $r = 115$, we have $z = 3$. Then, for $u = 3$, we have $s = 148$; for $u = 2$, we get $157 \leq s \leq 168$; and for $u = 1$, we have $175 \leq s \leq 226$. Our data show that, for $r = 115$, these are the only values of s for which the base case does not hold. We have verified similar results for other values of r up to 250. This leads us to make the following conjecture.

Conjecture 1. *For Type* II(a) *and for fixed r, the base case holds except for the ranges of values of s given by Theorem* 2.4.

This still leaves some open problems for not only Type II(a), but for others that follow:

Problem 1. For the pairs (m, n) for which the base case is known not to hold, determine the value of $\beta(m, n)$.

Problem 2. For the pairs (m, n) for which the base case is known not to hold, determine the value of $\beta(m, n)$.

We have some partial results, but they shed little light on the general problem.

2.2.2. Type II(b)

Having shown that the base case sometimes holds, we next show that in Type II(b), when $s \geq 2r+2$, the base case never holds. To this end, we introduce further notation that will also be used later. Let $s = (2r+1)q + t$, where q is the quotient and t the remainder when s is divided by $2r+1$ ($0 \leq t \leq 2r$). Note that $q \geq 1$, and if $q = 1$ then $t > 0$. We also let $\mu(m, n)$ denote the minimum of the two products that result when there is a shift of q from the base case in the partition of n; that is,

$$\mu(m, n) = \min\{r(s+1+q), (r+1)(s-q)\}.$$

Theorem 2.5. *If $m = 2r + 1$ and $n = 2s + 1$ with $s \geq 2r + 2$, then the base case does not hold; that is, $\beta(m, n) > r(s + 1)$.*

Proof. Let $s = (2r+1)q+t$ with $0 \leq t \leq 2r$. We observe that the difference of the products of the pairs $(r, s+1+q)$ and $(r+1, s-q)$ is $r-(s-2rq-q) = r-t$. Thus, $\mu(m, n) = r(s+1+q)$ for $r < t$ and $\mu(m, n) = (r+1)(s-q)$ for $r \geq t$. It is easily checked that in both cases this number is better than the base case. ■

In effect, what the proof of Theorem 2.5 establishes is that, for the values of m and n under consideration, $\beta(m, n) \geq \mu(m, n)$. We next show that equality holds here in some cases.

Theorem 2.6. *Let $m = 2r + 1$ and $n = 2s + 1$ with $1 \leq r \leq 8$ and $s \geq 2r + 2$. If $t \leq r$, then $\beta(m, n) = \mu(m, n)$.*

Proof. When $t \leq r$, we have $\mu(m, n) = (r + 1)(s - q) = 2qr^2 + (t + 2q)r + t$. Suppose there exist positive x and y such that $(r-x)(s+q+1+y) \geq \mu(m, n)+1$ and $(r+1+x)(s-q-y) \geq \mu(m, n)+1$. Adding, and simplifying, we get,

$$r(2s+1) + (s-q-y) - x(2q+2y+1) \geq 2(r+1)(s-q) + 2.$$

Further simplification gives

$$r \geq 2xq + 2xy + x + y + t + 2,$$

so that $r \geq 9$. ■

Table 3
Some values for Type III with m even and n odd.

r	s	$m = 2r$	$n = 2s + 1$	Base value $= rs$	$\beta(m, n)$
5	5	10	11	25	25
5	7	10	15	35	**36**
5	9	10	19	45	45
6	6	12	13	36	36
6	7	12	15	42	42
6	8	12	17	48	**49**
6	9	12	19	54	**55**

We conjecture that this result can be extended to higher values of r:

Conjecture 2. *If $1 \le r \le 15$, $m = 2r + 1$, $n = 2s + 1$, and $s \ge 2r + 2$, then $\beta(m, n) = \mu(m, n)$.*

We have verified our conjecture for $s \le 100{,}000$. Furthermore, if the conjecture is true, then it is sharp in that there are values of $r \ge 16$ for which the result does not hold. For example, if $r = 16$ and $s = 38$ (so $m = 33$ and $n = 77$), then $\mu(m, n) = \mu(33, 77) = 629$, while $\beta(33, 77) = 630$ with $(15, 42)$ and $(18, 35)$ as optimal pairs. Of course this is still greater than the base case value of 624.

2.3. Type III

Here things are not even this straightforward. As the bold-faced values of $\beta(m, n)$ for $m = 10$ and 12 in Table 3 show, the base case does not always hold. We note that for $m = 10$ and $n = 15$, there are two optimal pairs $(4, 9)$ and $(6, 6)$.

However, we show below that for a given value of m, the base case is optimal for all but a finite number of values of n.

Theorem 2.7. *Let $m = 2r$ and $n = 2s + 1$.*

(a) *If r is odd and $s \ge \frac{(r-1)^2}{2}$, then $\beta(m, n) = rs$.*

(b) *If r is even and $s \ge \frac{(r-1)(r-2)}{2}$, then $\beta(m, n) = rs$.*

Furthermore, both bounds are sharp.

Proof. We first show that if the base case does not hold (that is, if $\beta(m, n) > rs$), then $n \le r(r - 2)$.

Let q and t be the quotient and remainder when s is divided by r, so $s = qr + t$, with $0 \le t < r$ and $q \ge 1$. Suppose there is such a pair (m, n) for which the base case in not optimal. Then there exist x, y such that

$$(r - x)(s + y) \ge rs + 1 \tag{5}$$

and

$$(r + x)(s + 1 - y) \ge rs + 1. \tag{6}$$

It follows from (5) that $y \ge 1$ and $x \le r - 1$, and from (6) that $x \ge 1$ and $y \le s$. We next simplify and solve (5) and (6) for y to get

$$\frac{xs + 1}{r - x} \le y \le \frac{x(s + 1) + r - 1}{r + x}. \tag{7}$$

Substituting $s = qr + t$ and simplifying, we have

$$xq + \frac{x^2 q + xt + 1}{r - x} \le y \le xq + 1 + \frac{x(t - xq) - 1}{r + x}. \tag{8}$$

The first inequality of (8) implies that

$$y \ge xq + 1, \tag{9}$$

while the second yields

$$t > xq. \tag{10}$$

From (7) it follows that $(r+x)(xs+1) \le (r-x)(xs+x+r-1)$, which reduces to

$$(2s+1)x^2 \le r(r-2). \tag{11}$$

Consequently, $n \le r(r-2)$.

Now assume that r is odd. It follows that the base case holds if $n > r(r-2)$, that is, if $2s+1 \ge r(r-2)+1$. But since r is odd, this inequality is equivalent to $s \ge \frac{(r-1)^2}{2}$, which establishes (a).

Next, let r be an even number for which the base case does not hold, and suppose that $s \ge \frac{(r-1)(r-2)}{2}$. Then, as before, (5) through (11) hold. From (11) it follows that $x^2 \le \frac{r^2-r}{2s+1} \le \frac{r^2-2r}{r^2-3r+3} < 2$, so $x = 1$. Furthermore, from (8) it follows that $y \le q+1+\frac{t-q-1}{r+1} < q+2$, so that $y = q+1$.

The inequalities (8) now become

$$q + \frac{q+t+1}{r-1} \le q+1 \le q+1+\frac{t-q-1}{r+1}. \tag{8$'$}$$

The inequality on the left yields $q+t+1 \le r-1$, which, together with (10), implies that $q \le \frac{r-3}{2}$. Since r is even, it follows from the bound on s that $q = \frac{r-4}{2}$. Now (5) implies that $t \le \frac{r}{2}$, so that $s = qr+t \le \frac{r(r-4)}{2} + \frac{r}{2} = \frac{r(r-3)}{2}$, which contradicts our hypothesis on s and establishes (b).

We now show that the bounds on s are sharp. First, let r be odd and let $s = \frac{(r-1)^2}{2} - 1$. It can be easily checked that each of the products in the split with pairs $(r-1, \frac{r^2-r-2}{2})$ and $(r+1, \frac{r^2-3r+2}{2})$ is equal to $rs+1$. Hence $\beta(m,n) \ge rs+1$, so the base case does not hold.

Now, let r be even and let $s = \frac{(r-1)(r-2)}{2} - 1 = \frac{r(r-3)}{2}$. Then $q = \frac{r-4}{2}$ and $t = \frac{r}{2}$. Consider the split with pairs $(r-1, s+q+1)$ and $(r+1, s-q)$. It is straightforward to show that the product of the first is $rs+1$ and that of the second is $rs+2$. Hence, once again, $\beta(m,n) \ge rs+1$, and the base case does not hold. ■

For a given value of m and the finite numbers of values of n that do not satisfy the bounds in Theorem 3.1, we have data that show that $\beta(m,n)$ either equals the base case or is a value that follows certain formulas. Our data lead us to the conjecture below. As a reminder, the notation and restrictions are as follows:

$m = 2r$, $n = 2s+1$ with $r \le s$ and $s \le \frac{r^2-2r-1}{2}$ if r is odd and $s \le \frac{r(r-3)}{2}$ if r is even;

q and t are such that $s = qr+t$ with $0 \le t < r$ and $1 \le q \le \lfloor \frac{r-3}{2} \rfloor$.

Conjecture 3. 1. *If r is odd and $r \le s \le \frac{r^2-2r-1}{2}$, then*

$$\beta(m,n) = \begin{cases} (r+1)(s-q) & \text{if } q+1 \le t \le \dfrac{r-1}{2}, \\ (r-1)(s+q+1) & \text{if } \dfrac{r-1}{2} \le t \le r-q-2, \\ rs & \text{otherwise.} \end{cases}$$

2. *If r is even and $r \le s \le \frac{r(r-3)}{2}$, then*

$$\beta(m,n) = \begin{cases} (r+1)(s-q) & \text{if } q+1 \le t \le \dfrac{r-2}{2}, \\ (r-1)(s+q+1) & \text{if } \dfrac{r}{2} \le t \le r-q-2, \\ rs & \text{otherwise.} \end{cases}$$

We note that in the case of r odd, of the $\frac{r^2-4r+1}{2}$ values of s, the base case is conjectured to hold for $\frac{(r-1)^2}{4} - 2$ values and not to hold for $\frac{(r-3)^2}{4}$. Similarly, for r even, there are $\frac{r^2-5r+2}{2}$ values of s covered by the conjecture, with the base case believed to hold for $\frac{(r-2)^2}{4} - 2$ values of s and not to hold for $\frac{(r-3)^2-1}{4}$.

Table 4
Some values for Type IV with m odd and n even.

r	s	$m = 2r + 1$	$n = 2s$	Base value rs	$\beta(m, n)$
3	6	7	12	18	**20**
4	6	9	12	24	**25**
4	7	9	14	28	**30**
5	6	11	12	30	30
5	7	11	14	35	**36**
6	7	13	14	42	42

2.4. Type IV

As we will show, one reason why this case is interesting is because exact values have been hard to come by. Some examples are shown in Table 4. Following the format of earlier cases, we begin by fixing r and varying s, but then we see that there are times to do the opposite.

In the next two results we show that the only time that the base case holds (that is, where $\beta(m, n) = rs$) is when $r = s - 1$.

Theorem 2.8. $\beta(2r + 1, 2r + 2) = r(r + 1)$.

Proof. Suppose that there exist x and y for which the products $(r - x)(s + y)$ and $(r + x + 1)(s - y)$ both exceed rs. It follows from the first inequality that

$$y \geq \frac{xr + x + 1}{r - x} = x + \frac{x^2 - x + 1}{r - x}.$$

Since the second fraction is positive and y must be an integer, it follows that $y \geq x + 1$. On the other hand, the second implies that

$$y \leq \frac{xr + r + x}{r + x + 1} = x + 1 - \frac{x^2 + x + 1}{r + x + 1},$$

and from this we deduce, again since the fraction is positive, that $y \leq x$. Therefore, there cannot exist such x and y. ■

Theorem 2.9. *Let $m = 2r+1$ and $n = 2s$. If $s \geq r+2$, then the base case does not hold; that is, $\beta(2r+1, 2s) \geq rs+1$.*

Proof. All that needs to be done is to find one pair of numbers x and y for which both of the products $(r + x)(s - y)$ and $(r - x + 1)(s + y)$ exceed rs. Taking both x and y to be 1 suffices. With these values, we have

$$(r + x)(s - y) - rs = (r + 1)(s - 1) - rs = s - r - 1 \geq 1,$$

since $s \geq r + 2$. We also have

$$(r + 1 - x)(s + y) - rs = (r + 1)s - rs = r \geq 1,$$

which completes the proof. ■

It turns out that equality does not hold in general. We look next at some small values of r.

Theorem 2.10. *Let $m = 2r + 1$ and $n = 2s$ with $r < s$. Then for $r = 1, 2, 3$,*

(a) $\beta(3, 2s) = \lfloor \frac{4s}{3} \rfloor$;

(b)

$$\beta(5, 2s) = \begin{cases} 2\left\lfloor \dfrac{6s}{5} \right\rfloor + 1 & \textit{for } s \equiv 4 \ (\textit{mod } 5), \\ 2\left\lfloor \dfrac{6s}{5} \right\rfloor & \textit{otherwise;} \end{cases}$$

(c)

$$\beta(7,2s)=\begin{cases}3\left\lfloor\dfrac{8s}{7}\right\rfloor+1 & \textit{for } s\equiv 5\ (\textit{mod } 7),\\[1ex] 3\left\lfloor\dfrac{8s}{7}\right\rfloor+2 & \textit{for } s\equiv 6\ (\textit{mod } 7),\\[1ex] 3\left\lfloor\dfrac{8s}{7}\right\rfloor & \textit{otherwise}.\end{cases}$$

Proof. (a) Let $s = 3q + t$ with $0 \le t \le 2$. Then the split with pairs $(1, s + q)$ and $(2, s - q)$ shows that $\beta(3, 2s) \ge s + q = \lfloor \frac{4s}{3} \rfloor$. Suppose that equality does not hold. Then there exist x and y for which $(1 + x)(s - y) \ge s + q + 1$ and $(2 - x)(s + y) \ge s + q + 1$. It is easily shown that x must then be 1. It follows from the first inequality that $y \le q$ and from the second that $y \ge q + 1$, a contradiction.

(b) Let $s = 5q + t$ with $0 \le t \le 4$. For $0 \le t \le 3$, the split with pairs $(2, s + q)$ and $(3, s - q)$ shows that $\beta(5, 2s) \ge 2s + 2q = r\lfloor \frac{6s}{5} \rfloor$. Now suppose that equality does not hold. Then there exist x and y for which $(2 + x)(s - y) \ge 2s + 2q + 1$ and $(2 - x)(s + y)2s + 2q + 1$. If $x = 2$, then the second inequality gives $s + y \ge 2s + 2q + 1$ or $y \ge s$, a contradiction. Hence $x = 1$. It follows from the first inequality that $y \le q$ and from the second that $y \ge q + 1$, another contradiction. Hence, equality for $t \ne 4$ holds. For $t = 4$, we use the split with pairs $(2, 6q + 5)$ and $(3, 4q + 3)$, and then the rest of the proof follows as before.

(c) The proof is similar to those of (a) and (b), and is therefore omitted. ■

We now go back to the other extreme, where r is close to (but less than) s.

Theorem 2.11. *Let $r < s$, and let $m = 2r + 1$ and $n = 2s$. Further, let $d = s - r$ and $k = \lfloor \frac{d}{2} \rfloor$. If $r \ge k(3k^2 + 1)$, then $\beta(m, n) = (r + k)(s - k)$.*

Proof. It is easily seen that for $r \ge k(2d - 1)$, $\min\{(r - k)(s + k), (r + k)(s + 1 - k)\} = (r - k)(s + k)$. Now assume that there exist x and y which give a better split of m and n, whence $(r - x)(s + y) \ge rs + rk - sk - k^2 + 1$ and $(r + x)(s + 1 - y) \ge rs + rk - sk - k^2 + 1$. Then algebra yields

$$y \ge x + \frac{x^2 - dx + dk - k^2 + 1}{r - x} \quad \text{and} \quad y \le x + 1 - \frac{x^2 + dx + dk - k^2 + 1}{r + x}.$$

Since $d = 2k$ or $d = 2k + 1$, it follows that both fractions are positive, which yields $y \ge x + 1$ and $y \le x$. This contradiction completes the proof. ■

We look next at values of s close to about $2r$. Our data indicate that here $\beta(2r + 1, 2s) = (r + 1)(s - 1)$.

Theorem 2.12. *Let $s \ge r+2$, $m = 2r+1$ and $n = 2s$. If $2r-8 \le s \le 2r+1$, then $\beta(m, n) = (r+1)(s-1)$ except that $\beta(21, 28) = 144$, $\beta(23, 30) = 169$, $\beta(25, 32) = 196$, $\beta(25, 34) = 209$, $\beta(27, 36) = 240$, and $\beta(29, 40) = 286$.*

Proof. Note that when $r + 2 \le s \le 2r + 1$, $r(s + 1) \ge (r + 1)(s - 1) \ge rs + 1$. Therefore, we assume that there exist x and y which give a better split of m and n; that is, $(s - x)(r + y) \ge (s - 1)(r + 1) + 1$ and $(s+x)(r+1-y) \ge (s-1)(r+1)+1$. Simplifying these, we get (a) $sy \ge rx+xy+(s-r)$ and (b) $(r+1)x \ge sy+xy-r$. From (a) we deduce that since $y \ge 1$, $xs \ge s + 1$, and hence $x \ge 2$. From (b) we get $(r - 1)y \ge xr$, and so $y \ge 2$ also. Moreover, from a combination of (a) and (b) we deduce that $2r - s \ge y(2x - 1)$, from which it follows that $x = y = 2$ since $-1 \le 2r - s \le 8$ by hypothesis.

Next let $x = y = 2$, and let $k = 2r - s$. It follows that $k = 4, 5$, or 6, and (a) and (b) reduce to $k+4 \le r \le 2(k-1)$, so that $10 \le r \le 14$. Therefore there are just six possible pairs (r, s), and the values of β for these pairs follow by direct calculation. ■

We observe that not only does the conjecture of Gutiérrez and Lladó [6] not hold, but there are values of m and n for which it is off by an arbitrarily large amount. In fact, we have the following result that shows that $\beta(m, n)$ can exceed the base case by nearly $\frac{1}{2}m^2$.

Theorem 2.13. *Let $m = 2r + 1$ and $n = 2s$. If $n = 2m(m - 2)$ (that is, $s = 4r^2 - 1$), then $\beta(m, n) = rs + 2r^2 - r$.*

Proof. The split $(r, s + 2r - 1)$ and $(r + 1, s - 2r + 1)$ clearly gives the two equal products of $rs + 2r^2 - r$. Suppose this is not optimal. Then there exist non-negative integers x and y such that

$$(r + x)(s + 2r - 1 - y) \geq rs + 2r^2 - r + 1 \quad \text{and} \quad (r + 1 - x)(s - 2r + 1 + y) \geq rs + 2r^2 - r + 1.$$

It follows that

$$\frac{sx + x - 2rx + 1}{r + 1 - x} \leq y \leq \frac{sx + 2rx - x - 1}{r + x}.$$

Simplification leads to $2sx(x - 1) + 2r + 1 \leq 0$, which is a contradiction, thus proving the theorem. ■

3. Summary

The main question considered in this paper is this: for two positive integers m and n, maximize the minimum of the two numbers ab and $(m - a)(n - b)$ over all integers a and b for which $0 \leq a \leq m$ and $0 \leq b \leq n$. That is, determine

$$\beta(m, n) = \max\{\min\{ab, (m - a)(n - b)\}\}.$$

The problem is divided into four cases, depending on the parities of m and n, with $m \leq n$: Types I and II: Both have the same parity; and Types III and IV: One is even and the other is odd. In each case, there is a natural lower bound, resulting from splitting m and n into two nearly equal parts. This is called the base case, and sometimes, even when the exact value of $\beta(m, n)$ has not been found, we are able to show that it exceeds the base case.

Type I: $m = 2r$ and $n = 2s$, and the base case is the pair (r, s) with value rs.

Here, the base case always holds: $\beta(m, n) = rs$.

Type II: $m = 2r + 1$ and $n = 2s + 1$ with $r \leq s$, and the base case is the pair $(r, s + 1)$ with value $r(s + 1)$.
(a) $s \leq 2r + 1$. The base case holds for all $s = r, r + 1, \ldots, r + 6$ and $s = 2r - 3, 2r - 2, \ldots, 2r + 1$. Most of the other cases remain undetermined except for values found by computer.
(b) $s \geq 2r + 2$. The base case never holds. The exact value is known in some instances.

Type III: $m = 2r$ and $n = 2s + 1$, the base case is again the pair (r, s) with value rs.

The base case holds if r is odd and $s \geq \frac{(r-1)^2}{2}$ or if r is even and $s \geq \frac{(r-1)(r-2)}{2}$, and these bounds are sharp.

Type IV: $m = 2r + 1$ and $n = 2s$, the base case is again the pair (r, s) with value rs.
The base case holds only when $s = r + 1$.
As to the question of which pairs m and n does the base case hold, we have the following results for the four possible kinds of pairs.
Let m and n be positive integers with $m \leq n$.

- If m and n are both even, the base case always holds.
- If m and n are both odd, the base case never holds for $n \geq 2m + 3$.
- If m is even and n is odd, the base case always holds beyond some critical value n_0.
- If m is odd and n is even, the base case never holds for $n \geq m + 3$.

Thus, for any fixed m, the answer to the base case question is known for all but finitely many values of n. In loose terms, we can say that the base case holds about half the time, depending on the parity of the smaller of the two numbers.

Since we began the paper with a discussion of line graphs and the line completion number of a graph, it seems appropriate to conclude with the following theorem on the line completion number of complete bipartite graphs, focusing on what eventually happens when the size of the smaller partite set is fixed. (Recall that $\mathrm{lc}(K_{m,n}) = \beta(m, n) + 1$.)

Theorem 3.1. *The line completion number* $\mathrm{lc}(K_{m,n})$ *of the complete m-by-n bipartite graph with m fixed and* $m \leq n$ *satisfies the following:*

(a) *For m even,*
  (i) *if n is also even, then* $\mathrm{lc}(K_{m,n}) = \frac{1}{4}mn + 1$ *for all n.*
  (ii) *if n is odd, then* $\mathrm{lc}(K_{m,n}) = \frac{1}{4}m(n-1) + 1$ *for*

$$\begin{cases} n \geq \dfrac{1}{4}(m^2 - 6m + 12) & \textit{if } m \equiv 0 \pmod 4, \\ n \geq \dfrac{1}{4}(m^2 - 4m + 8) & \textit{if } m \equiv 2 \pmod 4. \end{cases}$$

(b) *For m odd,*
  (i) *if n is even, then* $\mathrm{lc}(K_{m,n}) \geq \frac{1}{4}n(m-1) + 2$ *for all* $n \geq m+3$.
  (ii) *if n is also odd, then* $\mathrm{lc}(K_{m,n}) \geq \frac{1}{4}(m-1)(n+1) + 2$ *for all* $n \geq 2m+3$.

References

[1] K.S. Bagga, L.W. Beineke, B.N. Varma, Super line graphs, in: Y. Alavi, A. Schwenk (Eds.), Graph Theory, Combinatorics, and Applications: Proceedings of the Seventh Quadrennial International Conference on the Theory and Applications of Graphs, vol. 1, Wiley, 1995, pp. 35–46.
[2] J.S. Bagga, L.W. Beineke, B.N. Varma, The super line graph $\mathcal{L}_2$, Discrete Math. 206 (1999) 51–61.
[3] J.S. Bagga, L.W. Beineke, B.N. Varma, Independence and cycles in super line graphs, Australas. J. Combin. 19 (1999) 171–178.
[4] X. Li, H. Li, H. Zhang, Path-comprehensive and vertex-pancyclic properties of super line graph $\mathcal{L}_2(G)$, Discrete Math. 308 (2008) 6308–6315.
[5] K.S. Bagga, L.W. Beineke, B.N. Varma, The line completion number of a graph, in: Y. Alavi, A. Schwenk (Eds.), Graph Theory, Combinatorics, and Applications: Proceedings of the Seventh Quadrennial International Conference on the Theory and Applications of Graphs, vol. 2, Wiley, 1995, pp. 1197–1201.
[6] A. Gutiérrez, A.S. Lladó, On the edge-residual number and the line completion number of a graph, Ars Combin. 63 (2002) 65–74.